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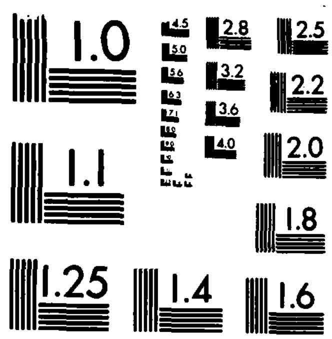

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FIXED POINT METHODS FOR THE COMPLEMENTARITY PROBLEM

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UNIVERSITY OF WISCONSIN-MADISON
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ABSTRACT

This paper is concerned with iterative procedures for the monotone complementarity problem. ~~Our~~ These iterative methods consist of finding fixed points of appropriate continuous maps. In the case of the linear complementarity problem, it is shown that the problem is solvable if and only if the sequence of iterates is bounded in which case summability methods are used to find a solution of the problem. This procedure is then used to find a solution of the nonlinear complementarity problem satisfying certain regularity conditions for which the problem has a nonempty bounded solution set. Keywords: monotone operators; operators (mathematics).

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SIGNIFICANCE AND EXPLANATION

Fixed point iterative procedures are defined which use summability theory for the solution of the monotone complementarity problem. Mangasarian & McLinden have established the existence and boundedness of solution set under certain regularity conditions. In this paper we provide a constructive procedure to obtain a solution under these conditions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

FIXED POINT METHODS

FOR THE COMPLEMENTARITY PROBLEM

P. K. Subramanian

1. Introduction

We are concerned in this paper with the *complementarity problem*, viz., that of finding a z_o (if it exists) such that $F(z_o) \geq 0$ and such that $z_o^T F(z_o) = 0$. Here F is an operator from $\Re^n$ to $\Re^n$. In particular, we are concerned with the case when F is *monotone*, that is

$$(x - y)^T \big(F(x) - F(y)\big) \geq 0 \quad \forall\, x, y \in \Re^n.$$

The operator F is *strongly monotone* if there exists a positive real number λ such that

$$(x - y)^T \big(F(x) - F(y)\big) \geq \lambda \|x - y\|^2.$$

When F is an affine map, $F(x) = Mx+q$, we shall refer to the complementarity problem as the *linear complementarity problem* and write $LCP(M, q)$ in this case. Otherwise we shall refer to the complementarity problem as the

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In the case of $LCP(M, q)$, when M is positive semidefinite, if the problem is feasible, that is there exists $x \geq 0$ such that $Mx + q \geq 0$, the problem is solvable [Eaves, 1971]. This is not the case for $NLCP(F)$ ([Megiddo, 1977], [Garcia, 1977]). However, for $\varepsilon > 0$, if we consider the *Tihonov regularization* $F_\varepsilon := F + \varepsilon I$ then the corresponding problem $NLCP(F_\varepsilon)$ has a unique solution since F_ε is strongly monotone [Karamardian, 1972]. When $\varepsilon \to 0$, then x_ε, the solution of $NLCP(F_\varepsilon)$, converges to the least two-norm solution of $NLCP(F)$, provided $NLCP(F)$ is solvable [Brézis, 1973].

A solution of $NLCP(F)$ is also a fixed point of the map

$$x \longmapsto \left(x - F(x)\right)_+ := \max\{0, x - F(x)\}.$$

The principal aim of this paper is to consider iterative procedures to find such fixed points. We shall show that in the linear case the sequence of iterates is bounded if and only if $LCP(M, q)$ is solvable. When this is the case, we use summability methods to obtain a solution of the problem. Although feasibility of the monotone $NLCP(F)$ does not imply its solvability, it is a theorem of Mangasarian & McLinden [1985] that when a regularity condition such as the *distribute Slater constraint qualification* is satisfied then the solution set is bounded. We show how the iterative procedure for the linear case may be adapted to find a solution in this special case.

We briefly describe the notation used in this paper. We use $\Re^n$ for the space of real ordered n-tuples. All vectors are column vectors and we use the Euclidean norm throughout. Given a vector x, we denote its i^{th} component

by x_i. We say $x \geq 0$ if $x_i \geq 0 \, \forall i$. The nonnegative orthant is denoted by $\Re^n_+$.

We use superscripts to distinguish between vectors, e.g., x^1, x^2 etc. For $x, y \in \Re^n$ x^T indicates the transpose of x, $x^T y$ their inner product. Occasionally, the superscript T will be suppressed. All matrices are indicated by upper case letters A, B, C etc. The i^{th} row of A is denoted by A_i while its j^{th}column is denoted by $A_{.j}$. The transpose of A is denoted by A^T.

Given $NLCP(F)$, we define the *feasible set* and *solution set* by $S(F)$ and $\overline{S}(F)$ respectively, that is,

$$S(F) = \{x \in \Re^n_+ : F(x) \in \Re^n_+\}$$

$$\overline{S}(F) = \{x \in S(F) \, : \, x^T F(x) = 0\}.$$

In the case of $LCP(M, q)$, we shall denote these sets by $S(M, q)$ and $\overline{S}(M, q)$ respectively. Finally the end of a proof is signified by ▌.

2. Fixed point methods

We begin with the well known notion of a *contraction mapping*.

2.1 Definition. *Let $P : D \subseteq \Re^n \longrightarrow \Re^n$. We say P is Lipschitzian with modulus $L > 0$ if*

$$\|P(x) - P(y)\| \leq L \|x - y\| \qquad \forall x, \, y \in D.$$

When $L \leq 1$ $(L < 1)$ we say P is non-expansive (contractive) .

The following Theorem is classical; see e.g., [Ortega and Rheinboldt, 1970, page 120].

2.2 Theorem. *(Banach's contraction mapping princple). Let P: $D \subseteq \Re^n \longrightarrow \Re^n$ and let D_0 be a closed subset of D such that $PD_0 = \{P(x) : x \in D_0\} \subseteq D_0$. If P is a contraction mapping on D_0 with modulus L, then P has a unique fixed point $\bar{x}$ in D_0. Further, for any point x^0 in D_0, the sequence $\{x^k\}$ where $x^{k+1} = P(x^k)$, converges to $\bar{x}$ with the following linear rate :*

$$\frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} \le L.$$

The content of the following proposition is well known. We state it in following form for later use and furnish a proof for the sake of completeness.

2.3 Proposition. *Let F: $D \subseteq \Re^n \longrightarrow \Re^n$ be monotone and Lipschitzian with modulus L. Suppose that $\varepsilon > 0$, $\alpha > 0$ and $\varepsilon\alpha \le 1$. Then the projection map* $\mathbb{P}$ defined by

$$\mathbb{P}(x) = \Big\{x - \alpha\big(F(x) + \varepsilon x\big)\Big\}_+, \qquad x \in D$$

is also Lipschitzian with modulus $k(\alpha) = \sqrt{(1-\alpha\varepsilon)^2 + (\alpha L)^2}$. If $\alpha < 2\varepsilon/\sqrt{\varepsilon^2 + L^2}$, then $\mathbb{P}$ is a contraction and k attains its minimum value

$$k_{\min}(\alpha) = L/\sqrt{L^2 + \varepsilon^2} \quad \textit{for} \quad \alpha = \varepsilon/L^2 + \varepsilon^2.$$

Proof

We have

$$\begin{aligned}\|\mathbb{P}(x) - \mathbb{P}(y)\|^2 &= \|\{x - \alpha(F(x) + \varepsilon x)\}_+ - \{y - \alpha(F(y) + \varepsilon y)\}_+\|^2 \\ &\le \|\{x - \alpha(F(x) + \varepsilon x)\} - \{y - \alpha(F(y) + \varepsilon y)\}\|^2\end{aligned}$$

since projection on $\Re^n_+$ is non-expansive. Hence,

$$\|\mathbb{P}(x) - \mathbb{P}(y)\|^2 \le \|(x-y)(1-\varepsilon\alpha) - \alpha\big(F(x) - F(y)\big)\|^2$$
$$= \|x-y\|^2 (1-\alpha\varepsilon)^2 + \alpha^2 \|F(x) - F(y)\|^2$$
$$- 2\alpha(1-\alpha\varepsilon)(x-y)\big(F(x) - F(y)\big).$$

Since $\alpha\varepsilon \le 1$ and $< F(x) - F(y),\, x - y > \ge 0$ from the monotonicity of F,

$$\|\mathbb{P}(x) - \mathbb{P}(y)\|^2 \le \|x-y\|^2 \left\{(1-\alpha\varepsilon)^2 + (\alpha L)^2\right\}.$$

The other claims about $k(\alpha)$ are easy to verify. ▮

2.4 Theorem. *Let $F : \Re^n \longrightarrow \Re^n$ be monotone and Lipschitzian with modulus L. Let $\{\varepsilon_n\}$ be a sequence of positive reals, $\varepsilon_n \downarrow 0$. For $n = 1, 2, \cdots$ let*

$$\mathbb{P}_n(x) = \big\{x - \alpha_n\big(F(x) + \varepsilon_n x\big)\big\}_+$$

and for $m = 1, 2, \cdots$ and $x \in \Re^n$ let

$$\mathbb{P}_n^m(x) = \underbrace{\mathbb{P}_n \circ \cdots \circ \mathbb{P}_n}_{m \text{ times}}(x) = x(n,m).$$

Suppose further that

$$\alpha_n = \frac{\varepsilon_n}{\varepsilon_n^2 + L^2}, \quad k_n = \frac{L}{\sqrt{L^2 + \varepsilon_n^2}}, \quad \delta_n = \varepsilon_n(1 - k_n).$$

For $n = 1, 2, \cdots,$ let $\bar{x}^n$ be defined by

$$\bar{x}^n = x(n,m), \quad \textit{where } \|x(n,m+1) - x(n,m)\| < \delta_n.$$

Then the sequence $\{\|\overline{x}^n\|\}$ is bounded if and only if $NLCP(F)$ is solvable and in this case, $\overline{x}_n \rightarrow \overline{x}$, the least two-norm solution of $NLCP(F)$.

Proof

From Proposition 2.3, $\mathbb{P}_n$ is a contraction with modulus $k_n < 1$. By the contraction mapping principle, given any x^0.

$$\lim_{j \to \infty} \mathbb{P}_n{}^j \longrightarrow z^n, \quad \mathbb{P}_n(z^n) = z^n.$$

Note that z^n solves $NLCP(F + \varepsilon_n I)$ uniquely. Since, by definition,

$$\mathbb{P}_n\big(x(n,m)\big) = x(n,\, m+1)$$

we have

$$\delta_n > \|x(n,m+1) - x(n,m)\| \geq \|x(n,m) - z^n\| - \|x(n,m+1) - z^n\|$$

and

$$\|x(n,m+1) - z^n\| = \|\mathbb{P}_n(x(n,m) - \mathbb{P}_n(z^n))\| \leq k_n \cdot \|x(n,m) - z^n\|$$

it follows that

$$\delta_n > \|x(n,m+1) - x(n,m)\| \geq (1 - k_n)\, \|x(n,m) - z^n\|$$

and that

$$\|\overline{x}^n - z^n\| < \varepsilon_n.$$

The conclusions about $\{\bar{x}^n\}$ follow from [Brézis, 1973] (see also [Subramanian, 1985]). ∎

We remark that the last Theorem is a *two-step* process in the sense that for a given ε_n, the contraction $\mathbb{P}_n$ is iterated m times until $x(n,m)$ is close enough to the solution z^n of $NLCP(F+\varepsilon_n)$. One then takes a smaller ε_n and the process repeats. Our aim now is to prove convergence for an algorithm which combines both steps into a single step. We shall need the following notions from the theory of summability.

2.5 Definition. *An infinite matrix $A = (A_{ij})$, i, $j = 1, 2, \cdots$, is said to be convergence preserving if for any sequence $\{x_n\}$, the sequence $\{y_n\}$ defined by*

$$y_n = \sum_{j=1}^{\infty} A_{nj} x_j \tag{2.6}$$

is well defined and $\lim x_n = \lim y_n$. We call $\{y_n\}$ the A-transform of $\{x_n\}$ and write $y_n = A(\{x_n\})$.

The following Theorem is classical. Its proof may be found for instance in [Peyerimhoff, 1969].

2.7 Theorem. *(O. Toeptlitz). An infinite matrix $A = (A_{ij})$, i, $j = 1, 2, \cdots$ is convergence preserving if and only if*

(1) $\sum_{j=1}^{\infty} |A_{ij}| = \sigma_i$ *exists,*

$\{\sigma_i\}$ *is bounded,*

(2) $\lim_i \left(\sum_{j=1}^{\infty} A_{ij}\right) = 1$,

(3) $\lim_i A_{ij} = 0$.

We are now ready to prove the principal theorem of this paper.

2.8 Theorem. *Let M be a positive semidefinite matrix. Assume that that the sequences $\{\overline{\alpha}_n\}$ and $\{\overline{\varepsilon}_n\}$ of positive reals are such that*

$$\left.\begin{array}{l} \displaystyle\sum_{n=1}^{\infty} \overline{\alpha}_n \ \ \textit{diverges,} \\ \displaystyle\sum_{n=1}^{\infty} \overline{\alpha}_n^2 \ \ \textit{converges,} \\ \displaystyle\sum_{n=1}^{\infty} \overline{\alpha}_n \overline{\varepsilon}_n \ \ \textit{converges and} \\ \overline{\varepsilon}_n \le 1, \quad \overline{\rho}_n = \dfrac{\overline{\alpha}_n}{\overline{\varepsilon}_n} \downarrow 0. \end{array}\right\} \tag{2.9}$$

Suppose that k is the smallest positive integer satisfying

$$\sqrt{2\overline{\rho}_k} + 2\overline{\rho}_k < L := \frac{1}{1 + \|M\|} \qquad (2.10).$$

Let $B = (B_{ij})$ be the infinite matrix whose n^{th} row B_n is defined by

$$B_n = \left(\frac{\overline{\alpha}_{1+k}}{S_n}, \ \frac{\overline{\alpha}_{2+k}}{S_n}, \dots, \ \frac{\overline{\alpha}_{n+k}}{S_n}, \ 0, \cdots \right)$$

where $S_n = \sum_{j=1}^{n} \overline{\alpha}_{j+k}$. Let $x^0 = 0$ and having x^n, determine x^{n+1} from

$$x^{n+1} = \left\{ (1 - \overline{\alpha}_{n+k}\overline{\varepsilon}_{n+k}) x^n - \overline{\alpha}_{n+k}(Mx^n + q) \right\}_+ \qquad (2.11).$$

Let $\{y^n\}$ be the B-transform of $\{x^n\}$, that is

$$y^n = \frac{1}{S_n} \left(\sum_{j=1}^{n} \overline{\alpha}_{j+k} x^j \right) \qquad (2.12).$$

Then $\overline{S}(M, q) \ne \emptyset \iff \{x^n\}$ is bounded. When this condition holds,

$$y^n \longrightarrow y^\star \in \overline{S}(M, q).$$

Proof

Assume that k satisfies (2.10). For notational convenience, we shall write

$$\alpha_n = \overline{\alpha}_{n+k}, \quad \varepsilon_n = \overline{\varepsilon}_{n+k} \quad \text{and} \quad \rho_n = \bar{\rho}_{n+k}.$$

Obviuosly, the sequences $\{\alpha_n\}$, $\{\varepsilon_n\}$ and $\{\rho_n\}$ also satisfy the conditions (2.13). We shall write

$$Fx = Mx + q, \; F_n x = Fx + \varepsilon_n x.$$

Thus we can write (2.11) in the form

$$x^{n+1} = \left\{(1 - \alpha_n \varepsilon_n)x^n - \alpha_n F x^n\right\}_+ \qquad (2.13).$$

We first assume that $\{x^n\}$ is bounded and show that in this case $y^n \to y^* \in \overline{S}(M, q)$ so that $\overline{S}(M, q) \neq \emptyset$.

Assume then that $\{x^n\}$ is bounded. Clearly, $\exists K_1 > 0$ and $K_2 > 0$ such that

$$\begin{aligned} \|x^n\| &\leq K_1, \\ \|F_n x^n\| &= \|Mx^n + q + \varepsilon_n x^n\| \\ &\leq (1 + \|M\|) \cdot \|x^n\| + \|q\| \\ &\leq (1 + \|M\|) \cdot K_1 + \|q\| \\ &:= K_2. \end{aligned}$$

Let $x \in \Re^n_+$ be arbitrary but fixed. Then from (2.13) we have

$$\begin{aligned}\|x^{n+1} - x\|^2 &= \|(x^n - \alpha_n(F_n x^n))_+ - x\|^2 \\ &\leq \|(x^n - x) - \alpha_n(Fx^n + \varepsilon_n x^n)\|^2 \\ &\leq \|(x^n - x)\|^2 - 2\alpha_n(Fx^n)(x^n - x) \\ &\quad - 2\alpha_n\varepsilon_n x^n(x^n - x) + \alpha_n^2 K_2^2. \end{aligned} \tag{2.14}$$

Since M is positive semidefinite we also have

$$(Fx^n)(x^n - x) \;\geq\; (Fx)(x^n - x).$$

Let

$$K_3 = \sup_n \|x^n\| \cdot \|x^n - x\|.$$

From (2.14) we now get

$$2\alpha_n(Fx)(x^n - x) \;\leq\; \|x^n - x\|^2 - \|x^{n+1} - x\|^2 + 2\alpha_n\varepsilon_n K_3 + \alpha_n^2 K_2^2.$$

Summing this from 1 to k we obtain

$$2(Fx)\sum_{n=1}^{k} \alpha_n(x^n - x) \;\leq\; \|x^1 - x\|^2 - \|x^{k+1} - x\|^2 + 2K_3\sum_{n=1}^{k}\alpha_n\varepsilon_n + K_2^2\sum_{n=1}^{k}\alpha_n^2.$$

Divide this last inequality by S_k and let $k \to \infty$. From the assumed properties in (2.9) of the sequences $\{\alpha_n\}$, $\{\varepsilon_n\}$ and from the definition of $\{y^n\}$, we now have

$$\liminf_k < Fx,\; x - y^k > \;\geq\; 0.$$

Since y^n is a convex combination of $x^1,\ x^2,\cdots,x^n$ it follows that $\{x^n\}$ bounded $\Rightarrow \{y^n\}$ is bounded. Hence $\{y^n\}$ has a limit point y^* for which

$$< Fx,\ x - y^* > \ \geq\ 0.$$

Since $x \in \Re^n_+$ was arbitrary, y^* solves $LCP(M,\,q)$. This completes our proof that

$$\{x^n\} \quad \text{bounded} \quad \Longrightarrow \quad \overline{S}(M,\,q) \neq \emptyset.$$

Next we prove $y^n \longrightarrow y^*$.

Since $\overline{S}(M,\,q) \neq \emptyset$, choose $z \in \overline{S}(M,\,q)$ arbitrary but fixed. By Theorem 2.4.2, z satisfies

$$< Fx^n,\ x^n - z > \ \geq\ 0 \tag{2.15}$$

since $x^n \geq 0$. From (2.13) and (2.15) we have

$$\begin{aligned} \|x^{n+1} - z\|^2 &\leq \|(x^n - z) - \alpha_n(Fx^n + \varepsilon_n x^n)\|^2 \\ &\leq \|x^n - z\|^2 - 2\alpha_n(Fx^n)(x^n - z) \\ &\qquad - 2\alpha_n\varepsilon_n\, x^n(x^n - z) + \alpha_n^2 K_2^2 \\ &\leq \|x^n - z\|^2 + 2\alpha_n\varepsilon_n\, |x^n(x^n - z)| + \alpha_n^2 K_2^2. \end{aligned}$$

Define $\beta_n(z)$ by

$$\beta_n(z) := 2\alpha_n\varepsilon_n\, \|x^n\|\, \|x^n - z\| + \alpha_n^2\, K_2^2 \tag{2.16}$$

and we now have

$$\|x^{n+1} - z\|^2 \ \leq\ \|x^n - z\|^2 + \beta_n(z) \tag{2.17}.$$

Let $\overline{S}$ denote $\overline{S}(M, q)$ and let

$$z^n = P_{\overline{S}}(x^n).$$

We are going to show that $\exists z^*$ such that

$$z^n \longrightarrow z^*, \quad y^n \longrightarrow z^*.$$

From (2.17) and the definition of z^n,

$$\|x^{n+1} - z^{n+1}\|^2 \leq \|x^{n+1} - z^n\|^2 \leq \|x^n - z^n\|^2 + \beta_n(z).$$

Since $\sum_n \beta_n(z)$ converges, by [Cheng, 1981, Lemma 2.2.12], we can conclude that

$$\|x^n - z^n\| \quad \text{converges.} \tag{2.18}$$

By parallelogram law, for $m > 0$,

$$\|z^{n+m} - z^n\|^2 = 2\|x^{n+m} - z^n\|^2 + 2\|x^{n+m} - z^{n+m}\|^2$$
$$- 4\|x^{n+m} - \frac{1}{2}(z^n + z^{n+m})\|^2.$$

Since $\overline{S}$ is convex, $(z^n + z^{n+m})/2 \in \overline{S}$. Also, z^{n+m} is the closest point to x^{n+m} in $\overline{S}$. Hence,

$$\|z^{n+m} - z^n\|^2 \leq 2\|x^{n+m} - z^n\|^2 - -2\|x^{n+m} - z^{n+m}\|^2. \tag{2.19}$$

Letting $z = z^n$ in (2.17) and noting that z^n is the closest point to x^n in $\overline{S}$, it follows that $\beta_n(z^n) \leq \beta_n(z)$. Now let $z = z^n$ in (2.17) and use induction to get

$$\|x^{n+m} - z^n\|^2 \leq \|x^n - z^n\|^2 + \sum_{j=n}^{n+m} \beta_j(z), \qquad m > 0.$$

Substitute this in (2.19) and we have

$$\|z^{n+m} - z^n\|^2 \le 2\|x^n - z^n\|^2 - 2\|x^{n+m} - z^{n+m}\|^2 + 2\sum_{j=n}^{n+m} \beta_j(z). \tag{2.20}$$

From (2.18) and the fact $\sum_n \beta_n(z)$ converges, we have by letting $n, m \to \infty$ in (2.20) that

$$\|z^{n+m} - z^n\| \longrightarrow 0$$

so that $\{z^n\}$ is Cauchy. Since $\overline{S}$ is closed, $\exists\, z^* \in \overline{S}$ such that $z^n \to z^*$. We shall now show that $y^n \to z^*$ as well.

Since $\{y^n\}$ is also bounded, let y^* be any of its limit points. Assume that the subsequence y^{n_k} converges to y^*. From our proof earlier, $y^* \in \overline{S}$. Observe that

$$z^j = P_{\overline{S}}(x^j) \Longrightarrow \langle x^j - z^j, y^* - z \rangle \le 0. \tag{2.21}$$

Multiply (2.21) by α_j^2 and sum from $j = 1, 2, \cdots, n_k$ to get

$$\left\langle \sum_{j=1}^{n_k} \alpha_j (x^j - z^j), \sum_{j=1}^{n_k} \alpha_j (y^* - z^j) \right\rangle \le 0.$$

Divide the last inequality by $S_{n_k}^2$ to obtain

$$\left\langle y^{n_k} - \frac{1}{S_{n_k}} \sum_{j=1}^{n_k} \alpha_j z^j,\ y^* - \frac{1}{S_{n_k}} \sum_{j=1}^{n_k} \alpha_j z^j \right\rangle \le 0. \tag{2.22}$$

Notice however that

$$\xi^{n_k} := \frac{1}{S_{n_k}} \left(\sum_{j=1}^{n_k} \alpha_j z^j \right)$$

is simply a subsequence of the B-transform of $\{z^n\}$, that is of

$$\{\xi^n\} = B(\{z^n\}).$$

Since B satisfies all the conditions of Theorem 2.7, it is a convergence preserving matrix. However, $z^n \to z^*$ so that both ξ^n and ξ^{n_k} also converge to z^*. If we take limits as $k \to \infty$ in (2.22), we get

$$< y^* - z^*, \; y^* - z^* > \leq 0$$

so that $y^* = z^*$. But y^* was any *arbitrary* limit point of $\{y^n\}$. Hence $y^n \to z^*$. This completes our proof that

$$\{x^n\} \text{ bounded} \quad \Longrightarrow \quad \overline{S}(M, q) \neq \emptyset \text{ and } y^n \longrightarrow z^* \in \overline{S}(M, q).$$

We now prove the converse, that is we shall assume that $\overline{S}(M, q) \neq \emptyset$ and show that $\{x^n\}$ is bounded.

Recall from (2.10) that k satisfies

$$\sqrt{2\bar{\rho}_k} + 2\bar{\rho}_k \;<\; L \;=\; \frac{1}{1 + \|M\|}.$$

Hence there exists σ, $0 < \sigma < 1/2$ for which

$$\sqrt{2\bar{\rho}_k} + \bar{\rho}_k \;<\; \frac{1}{(1+\sigma)(1 + \|M\|)} \;:=\; L_\sigma \quad .$$

The function $f(r)$,

$$f(r) := \frac{r}{r(1+\sigma)(1 + \|M\|) + \|q\|}$$

is strictly increasing in $[0, \infty]$, $\lim_r f(r) = L_\sigma$. Thus $\exists \bar{r} > 0$ such that for $r > \bar{r}$,

$$\sqrt{2\bar{\rho}_k} + 2\bar{\rho}_k < f(\bar{r}) < f(r).$$

Since $\bar{\rho}_n \downarrow 0$ and $\bar{\rho}_{n+k} = \rho_n$, we have for all $n > 0$ and $r > \bar{r}$,

$$\sqrt{2\rho_n} + 2\rho_n < f(\bar{r}) < f(r). \tag{2.23}$$

By assumption, $\overline{S} \neq \emptyset$. Let $z = P_{\overline{S}}(0)$, that is z is the least two-norm solution of $LCP(M, q)$. Define

$$r = \max(\bar{r}, \frac{1}{\sigma}\|z\|) + 1.$$

Our aim is to show that

$$\|x^n - z\| \leq r, \qquad \forall n \geq 0,$$

that is $\{x^n\}$ is bounded and this would complete our proof.

We use induction.

For $n = 0$, $\|x^0 - z\| = \|z\| < \sigma r < r$.

Suppose now that $\|x^n - z\| \leq r$. Let $\mu_n = \|x^n - z\|$. From (2.13),

$$\begin{aligned}\mu_{n+1}^2 &= \|x^{n+1} - z\|^2 \\ &\leq \|x^n - z\|^2 - 2\alpha_n\epsilon_n\, x^n(x^n - z) \\ &\quad - 2\alpha_n(Fx^n)(x^n - z) + \alpha_n^2\|Fx^n + \epsilon_n x^n\|^2.\end{aligned} \tag{2.24}$$

Since $z \in \overline{S}$, $(Fx^n)(x^n - z) \geq 0$. Also if $\mu_{n+1} \leq \mu_n$, we are done. So assume $\mu_{n+1} > \mu_n$. From (2.24) we thus get

$$2\alpha_n \varepsilon_n x^n(x^n - z) < \alpha_n^2 \|Fx^n + \varepsilon_n x^n\|^2,$$

that is

$$x^n(x^n - z) < \frac{\rho_n}{2} \|Fx^n + \varepsilon_n x^n\|^2. \tag{2.25}$$

Since

$$\begin{aligned} \|x^n\| &\leq \|x^n - z\| + \|z\| \\ &\leq r + \sigma r \\ &= (1 + \sigma)r, \end{aligned}$$

we have

$$\begin{aligned} \|Fx^n + \varepsilon_n x^n\| &\leq \|Mx^n + q\| + \varepsilon_n \|x^n\| \\ &\leq \|M\| \, \|x^n\| + \|q\| + \|x^n\| \\ &\leq (1 + \sigma)r \cdot (1 + \|M\|) + \|q\| \\ &= \frac{r}{f(r)} \\ &=: \xi \quad \text{say}. \end{aligned} \tag{2.26}$$

From (2.25) we now get

$$\begin{aligned} x^n(x^n - z) &< \frac{\rho_n}{2} \xi^2, \\ (x^n - z)(x^n - z) &< \frac{\rho_n}{2} \xi^2 - z(x^n - z) \\ &\leq \frac{\rho_n}{2} \xi^2 + \|z\| \cdot \|x^n - z\| \quad . \end{aligned}$$

Rewriting this last inequality,

$$\mu_n^2 \; < \; \frac{\rho_n}{2}\xi^2 + r\sigma\mu_n,$$

whence

$$2\mu_n^2 - 2r\sigma\mu_n - \rho_n\xi^2 \; < \; 0.$$

Since $\mu_n \; \geq \; 0$, we must have

$$\begin{aligned} \mu_n \; &< \; \frac{2r\sigma + \sqrt{4r^2\sigma^2 + 8\xi^2\rho_n}}{4} \\ &\leq \; \frac{r\sigma}{2} + \left(\frac{2r\sigma + 2\xi\sqrt{2\rho_n}}{4}\right) \end{aligned}$$

and finally

$$\mu_n \; < \; r\sigma + \frac{\xi}{2}\sqrt{2\rho_n}. \tag{2.27}$$

Again from the definition of x^{n+1} in (2.13),

$$\begin{aligned} \mu_{n+1} &= \|x^{n+1} - z\| \\ &\leq \; \|x^n - z - \alpha_n(F_n x^n)\| \\ &\leq \; \mu_n + \alpha_n\|F_n x^n\| \\ &\leq \; \mu_n + \rho_n\xi, \end{aligned} \tag{2.28}$$

where we have used (2.26) and the fact $\alpha_n \; < \; \rho_n$. If we use our estimate of μ_n from (2.27) in (2.28) we get

$$\mu_{n+1} \; < \; r\sigma + \frac{\xi}{2}\sqrt{2\rho_n} + \rho_n\xi.$$

Substituting for ξ from (2.26) and using (2.23) we finally get

$$\begin{aligned}\mu_{n+1} &< r\sigma + \frac{(\sqrt{2\rho_n} + 2\rho_n)}{2} \cdot \frac{r}{f(r)} \\ &< r\sigma + \frac{f(r)}{2} \cdot \frac{r}{f(r)} \\ &= r\sigma + \frac{r}{2} \\ &< r\end{aligned}$$

since $\sigma < 1/2$. Hence $\mu_{n+1} < r$. This completes our induction and also the proof of the Theorem. ∎

2.29 Remark

Our proof showing that $\{y^n\}$ converges by considering $z^n = P_{\overline{S}}(x^n)$ is patterned after [Baillon, 1975], who uses this technique to construct fixed points of non-expansive maps. Notice also Baillon's use of the *Cesàro matrix* C where $C_{ij} = 1/i$ for $j \leq i$, while $C_{ij} = 0$ for $j > i$.

3. Application to NLCP(F)

We shall now show that the proof of Theorem 2.8 can be used to construct a solution of $NLCP(F)$ when F is monotone and satisfies some regularity conditions such as the *distributed Slater constraint qualification* [Mangasarian and McLinden, 1985].

3.1 Definition. *Let F: $D \subseteq \Re^n \longrightarrow \Re^n$. We say that F satisfies the distributed Slater constrainst qualification (DSCQ) if there exist p points $z^1, z^2, \cdots,, z^p \in D$, nonnegative weights $\lambda_1, \lambda_2, \cdots \lambda_p$ $\left(\sum_j \lambda_j = 1\right)$ such that $\hat{z} = \sum_j \lambda_j z^j \geq 0$ and $\hat{w} = \sum_j \lambda_j w^j > 0$ where $w^j = F(z^j)$.*

Mangasarian and McLinden have proved the following Theorem.

3.2 Theorem. *Let $F\colon D \subseteq \Re^n \longrightarrow \Re^n$, $\Re^n_+ \subset D$ and suppose that F is monotone and continuous on D. Assume that F satisfies (DSCQ). Let*

$$\gamma > \max\left(1,\ -\hat{z}\hat{w} + \sum_{j=1}^{p} \lambda_j z^j w^j\right),$$

$$C = \{z \in \Re^n_+ : \hat{w}z \leq \hat{w}\hat{z} + \gamma\}$$

where λ_j, z^j, w^j, $\hat{z}$ and $\hat{w}$ are as in (DSCQ). Then $NLCP(F)$ is solvable and has a solution z^* such that $\hat{w}z^* < \hat{w}\hat{z} + \gamma$.

We shall now show that the technique used in the proof of Theorem 2.8 can be used to construct a solution of $NLCP(F)$ guaranteed by Theorem 3.2.

3.3 Theorem. *Assume that F satisfies the hypotheses of Theorem 3.2 and let C be the compact convex set as defined in that Theorem. Let $x^0 = 0$ and given x^n find from x^{n+1}*

$$x^{n+1} = P_C\{x^n - \frac{F(x^n)}{n}\}.$$

Let B be the Césaro matrix with

$$B_n = \left(\frac{1}{S_n}, \frac{1}{2S_n}, \cdots, \frac{1}{nS_n}, 0, 0, \cdots\right), \quad S_n = \sum_{j=1}^{n} \frac{1}{j}$$

and let $\{y^n\} = B(\{x^n\})$. Then y^n converges to a solution of $NLCP(F)$.

Proof

We shall give only a brief outline. Since $\{x^n\}$ and hence $\{y^n\}$ are both bounded, $\{y^n\}$ has a limit point y^* . One uses the monotonicity of F to

show that

$$\langle F(y^*, x - y^*\rangle \geq 0, \qquad \forall x \in C.$$

Hence y^* is a fixed point of the map $x \mapsto P_C(x - F(x))$. However, Mangasarian and McLinden show that any such fixed point satisfies $\hat{w}y^* < \hat{w}\hat{z} + \gamma$. Hence y^* solves $NLCP(F)$. One can now show that $y^n \to y^*$ by considering the projection z^n of x^n on $\bar{S}(F)$. ∎

3.4 Remarks

1. It is easy to see that Theorem 2.8 may be extended to the nonlinear case if F is Lipschitzian. In this case $\|M\|$ is replaced by the Lipschitz constant of F in (2.10).

2. Unfortunately, from a computational point of view the fixed point methods in general, and those considered in this paper in particular, are not viable methods. They are extremely slow and particularly so in the vicinity of a solution point since the step sizes taken in such a vicinity are extremely small. Their slowness in part is also due to the fact that they do not utilize special features of the matrix M in the case of $LCP(M, q)$. Their real utility is perhaps in generating good starting points for fast Newton-type algorithms. However, the SOR methods are much faster than the fixed point methods even for generation of starting points.

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This paper is concerned with iterative procedures for the monotone complementarity problem. Our iterative methods consist of finding fixed points of appropriate continuous maps. In the case of the linear complementarity problem, it is shown that the problem is solvable if and only if the sequence of iterates is bounded in which case summability methods are used to find a solution of the problem. This procedure is then used to find a solution of the nonlinear complementarity problem satisfying certain regularity conditions for which the problem has a nonempty bounded solution set.

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